General Relativity

Final Exam
30/10/2014
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Problem 1

Consider the 2-dimensional spacetime[1]

$$ds^2 = r^2 dt^2 - \frac{dr^2}{r^2}$$

defined for $r > 0$.

1. Write the components of the metric and compute the Christoffel symbols.
2. Consider the Riemann tensor with all components lower. How many independent components does it have in 2 dimensions? Compute *the independent components* of the Riemann tensor (with lower indices) for this metric.
3. Derive the equations of motion for massive particles moving in this spacetime.
4. Show that massive particles moving along geodesics can never reach the region $r \to \infty$.
5. Consider a massive particle of mass m staying at constant value of $r = r_0$. Compute the magnitude of relativistic acceleration that the particle undergoes and the force needed to keep it in this orbit.

Problem 2

Consider a massive test particle moving in the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

1. Derive the equations describing inertial motion of the particle

2. Find the radius of the smallest possible circular orbit. Argue that this orbit is unstable under small perturbations.

3. Find the radius of the smallest possible *stable* circular orbit.

Problem 3

Consider a Robertson-Walker metric with flat spatial section ($k = 0$):

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

[1] This is called 2-dimensional anti-de Sitter spacetime.

1. Suppose that the universe is filled with perfect fluid whose equation of state is $P = w\rho$, where w is some constant. Using the FRW equations

$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3P)$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho$$

Show that the scale factor a obeys the condition

$$\rho a^{3(1+w)} = C$$

where C is some constant.

2. Using this show that (if $w > -1$) the scale factor will evolve with time like

$$a(t) = D(t - t_0)^{\frac{2}{3(1+w)}}$$

where t_0 is an integration constant and D is another constant. At $t = t_0$ the scale factor goes to zero, so it is natural to identify $t = t_0$ with the Big Bang singularity.

3. Notice that the case $w = -1$ is special and the previous formula does not apply. Verify that the FRW equations with a fluid which has equation of state with $w = -1$ are identical to the equations that we would get without any fluid, but with non-vanishing cosmological constant Λ

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 0$$

Find the relation between the density ρ of the fluid with $w = -1$ and the effective cosmological constat parameter Λ.

4. In the case where we have fluid with $w = -1$ find the time-dependence of the scale factor $a(t)$.

Solution problem 1:

1. We have

$$g_{tt} = r^2 \qquad g_{rr} = -1/r^2 \qquad g_{tr} = g_{rt} = 0$$

$$\Gamma^t_{tt} = 0$$

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{r}$$

$$\Gamma^r_{tt} = r^3$$

$$\Gamma^r_{rt} = \Gamma^r_{tr} = 0$$

$$\Gamma^t_{rr} = 0$$

$$\Gamma^r_{rr} = -\frac{1}{r}$$

2.

$$R_{trtr} = \pm 1$$

3. Consider parametrizing the curve as $t(\tau), r(\tau)$. Minimizing the length in affine parametrization we find

$$r^2 \dot{t} = k$$

and the affine parametrization condition

$$r^2 \dot{t}^2 - \frac{\dot{r}^2}{r^2} = 1$$

or

$$\frac{k^2}{r^2} - \frac{\dot{r}^2}{r^2} = 1$$

or

$$\dot{r}^2 + r^2 = k^2$$

We notice that the effective potential goes like r^2 and is hence unbounded towards $r \to \infty$.

5. The orbit of the particle is $x^\mu = (t(\tau), r_0)$. We compute the 4-velocity as

$$u^\mu = (\frac{1}{\sqrt{r_0}}, 0)$$

The 4-acceleration is

$$a^\nu = u^\mu \nabla_\mu u^\nu = u^\mu \Gamma^\nu_{k\mu} u^k = \frac{1}{r_0} \Gamma^\nu_{tt}$$

so

$$a^t = 0 \qquad a^r = r_0^2$$

We compute the magnitude

$$g_{\mu\nu} a^\mu a^\nu = -r_0^2$$

Solution problem 2: In the usual way we find the equations of motion

$$\dot{t}\left(1 - \frac{2GM}{r}\right) = k$$

$$r^2\dot{\phi} = h$$
$$\left(1-\frac{2GM}{r}\right)\dot{t}^2 - \left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 1$$

Replacing the first two we find

$$\frac{k^2}{1-\frac{2GM}{r}} - \frac{\dot{r}^2}{1-\frac{2GM}{r}} - \frac{h^2}{r^2} = 1$$

or

$$k^2 - \dot{r}^2 - \frac{h^2}{r^2}\left(1-\frac{2GM}{r}\right) = 1 - \frac{2GM}{r}$$

Differentiating and eliminating $\dot{r}$ we find

$$-2\ddot{r} + 2\frac{h^2}{r^3} - \frac{6GMh^2}{r^4} = \frac{2GM}{r^2}$$

For $\ddot{r} = 0$ we find the condition

$$h^2 r - 3GMh^2 - GMr^2 = 0$$

or

$$r^2 - \frac{h^2}{GM}r + 3h^2 = 0$$

Solving this quadratic equation we find

$$r_\pm = \frac{h^2}{2GM} \pm \sqrt{\frac{h^4}{4G^2M^2} - 3}$$

Solution problem 3:

1. We have the two FRW equations:

$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3P) \tag{1}$$
$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho \tag{2}$$

Let us consider the time derivative of the combination $\rho a^{3(1+w)}$. We have

$$\frac{d}{dt}(\rho a^{3(1+w)}) = \dot{\rho} a^{3(1+w)} + 3(1+w)\rho\frac{\dot{a}}{a}a^{3(1+w)} \tag{3}$$

In order to compute $\dot{\rho}$ we differentiate (2) to find

$$\dot{\rho} = \frac{3}{8\pi G}\left(\frac{2\dot{a}\ddot{a}}{a^2} - 2\frac{\dot{a}^3}{a^3}\right)$$

We replace the first term in the parenthesis from (1) and the second from (2) to find

$$\dot{\rho} = -\rho\frac{\dot{a}}{a}3(1+w)$$

Replacing this in (3) we get the desired relation

$$\rho a^{3(1+w)} = C = \text{constant}$$

2. Let us assume that $w \neq -1$. Plugging the previous result in (2) we find

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\frac{C}{a^{3(1+w)}}$$

or

$$a^{3w+1}\dot{a}^2 = \frac{8\pi GC}{3}$$

or (assuming that we are in an expansion phase, so that $\dot{a} > 0$)

$$a^{\frac{3w+1}{2}}da = \frac{8\pi GC}{3}dt$$

or

$$\frac{2}{3(w+1)}a^{\frac{3(w+1)}{2}} = \frac{8\pi GC}{3}t + \text{constant}$$

We can rewrite the integration constant in the following form

$$a(t) = D(t-t_0)^{\frac{2}{3(1+w)}} \qquad , \qquad D = [4\pi GC(1+w)]^{\frac{2}{3(1+w)}}$$

3. The FRW equations in the presence of the fluid with $w = -1$ and absence of cosmological constant were derived from

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

where $T_{\mu\nu}$ was the perfect fluid stress tensor $T_{\mu\nu} = (\rho + P)u_\mu u_\nu - P g_{\mu\nu}$. For $w = -1$ we find $P = -\rho$ and hence

$$T_{\mu\nu} = \rho g_{\mu\nu}$$

Hence the FRW equations were derived from

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\rho g_{\mu\nu} \tag{4}$$

with ρ being a constant.

On the other hand, in the absence of any fluid, but in the presence of the cosmological constant, we have the equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 0$. Moving the last term to the RHS we get the equations in the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu}$$

These are identical to equations (4) if we idenify

$$\Lambda = 8\pi G\rho$$

4. If $w = -1$ we find from the first item that

$$\rho = \text{constant}$$

Hence from (2) we find that (assuming $\dot{a} > 0$)

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G\rho}{3}}$$

or

$$a(t) = a_0 e^{\sqrt{\frac{8\pi G\rho}{3}}t}$$

where a_0 is some constant.